Note: Fast Fourier Transform

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Convolution

Linear convolution:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt$$

Circular convolution (f and g are both T-periodic):

$$(f \circledast g)(x) = \int_{t_0}^{t_0+T} f(t)g(x-t)dt$$

If f is aperiodic and g is T-periodic, then:

$$(f * g)(x) = (f_T \circledast g)(x), \quad \text{where} \quad f_T(x) = \sum_{k=-\infty}^{\infty} f(x+kT)$$

Discrete Convolution

Linear convolution of two sequences $x[n]$ and $y[n]$:

$$(x * y)[n] = \sum_{m=-\infty}^{\infty} x[m]y[n-m]$$

Circular convolution ($x[n]$ and $y[n]$ are both N-periodic or of length N):

$$(x \circledast y)[n] = \sum_{m=0}^{N-1} x[m]y[n-m \mod N]$$

If $y[n]$ is N-periodic, then:

$$(x * y)[n] = (x_N \circledast y)[n], \quad \text{where} \quad x_N[n] = \sum_{k=-\infty}^{\infty} x[n-kN]$$

Fourier Transform

Fourier transform:

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}dx$$

Convolution theorem:

$$(f * g)(x) = \mathcal{F}^{-1}\big(\mathcal{F}(f) \cdot \mathcal{F}(g)\big)(x)$$

Discrete-time Fourier Transform

Let consider a continuous-time signal $x_c(t)$. For a given sampling period T, the sampling times are

$$t_n = nT, \quad n \in \mathbb{Z},$$

and the sampling values are

$$x[n] = x_c(t_n) = x_c(nT).$$

The discrete-time Fourier transform (DTFT) is defined as:

$$X_{1/T}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-2\pi i\omega nT}$$

Let define

$$x_d(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT),$$

then, the DTFT can be seen as the Fourier transform of $x_d(t)$:

$$\begin{aligned}
\mathcal{F}(x_d)(\omega) &= \int_{-\infty}^{\infty} x_d(t)e^{-2\pi i\omega t}dt \\
&= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t - nT)e^{-2\pi i\omega t}dt \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{-2\pi i\omega nT}.
\end{aligned}$$

Convolution theorem for DTFT. Consider two sequences $x[n]$ and $y[n]$ and their DTFTs $X_{1/T}(\omega)$ and $Y_{1/T}(\omega)$. Then:

Discrete Fourier Transform

The discrete Fourier transform (DFT) is defined as:

$$X_k = \sum_{n=0}^{N-1} x_N[n]e^{-2\pi i\frac{k}{N}n}, \quad k = 0, \dots, N-1,$$

where

$$x_N[n] = \sum_{m=-\infty}^{\infty} x[n - mN].$$

The DFT can be seen as the DTFT sampled at

$$\omega_k = \frac{k}{NT}.$$

We have:

$$\begin{aligned}
X_{1/T}(\omega_k) &= X_{1/T}(\frac{k}{NT}) \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{-2\pi i \frac{k}{N} n} \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=0}^{N-1} x[n-mN]e^{-2\pi i \frac{k}{N}(n-mN)} \\
&= \sum_{n=0}^{N-1} \sum_{m=-\infty}^{\infty} x[n-mN]e^{-2\pi i \frac{k}{N} n} \\
&= \sum_{n=0}^{N-1} x_N[n]e^{-2\pi i \frac{k}{N} n}
\end{aligned}$$