Block randomization

Jérôme Pasquier

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In the context of randomized block allocation between two treatments A and B, let $m \in \mathbb{N}^*$ be the block half-size and $T_k \in \{0, 1, \ldots, m\}$ be the number of patients allocated to treatment A after k asignments, where $k \in \{0, 1, \ldots, 2m\}$.

Knowning that $T_k = i$, the conditional probabilities for the next allocation are:

$$\Pr\big(T_{k+1} = i + 1 \mid T_k = i\big) = \frac{m - i}{2\,m - k},$$
$$\Pr\big(T_{k+1} = i \mid T_k = i\big) = \frac{m - k + i}{2\,m - k}.$$

The unconditional probabilities $p_k(i) = \Pr\big(T_k = i\big)$ can be defined recusively as follows:

$$p_{k+1}(i) = \frac{(m - i + 1)\,p_k(i-1) + (m - k + i)\,p_k(i)}{2\,m - k}$$

with $p_0(i) = \mathbf{1}\{i = 0\}$ and $p_k(-1) := 0$.

Note that $p_k(i) > 0$ only if $\max(0, k - m) \leq i \leq \min(k, m)$.

This recursion characterizes the hypergeometric distribution, such that

$$T_k \sim \text{Hypergeometric}(2m, m, k),$$

with

$$p_k(i) = \Pr(T_k = i) = \frac{\binom{m}{i}\binom{m}{k-i}}{\binom{2\,m}{k}}.$$

Properties.

- $p_k(i) = p_k(k - i)$ for $i \leq k \leq m$;
- TO SHOW: $p_{2m-k}(m - i) = p_k(i)$

Let $D_k = 2T_k - k$ be the signed difference between the numbers assigned to treatments A and B after k allocations.

The maximal absolute difference after k allocations is

$$\delta_k = \min(k, 2\,m - k).$$

The variable D_k takes values in

$$\mathcal{D}_k = \{-\delta_k, -\delta_k + 2, \ldots, \delta_k - 2, \delta_k\},$$

and is distributed as

$$q_k(d) = \Pr(D_k = d) = p_k\left(\frac{d + k}{2}\right).$$

The symmetry $p_k(i) = p_k(k - i)$ implies the symmetry $q_k(-d) = q_k(d)$.

1 Quantities of interest

1.1 Expected imbalance

The expected absolute imbalance after k allocations is given by

$$\mathbb{E}(|D_k|) = \sum_{i=0}^{m} |2i - k| \, p_k(i),$$

or, equivalently,

$$\mathbb{E}(|D_k|) = \sum_{d \in \mathcal{D}_k^+} 2d \, q_k(d).$$

1.2 Probability of a large imbalance

$$\Pr(|D_k| \geq d^*) = \sum_{\substack{d \in \mathcal{D}_k^+ \\ d \geq d^*}} 2q_k(d).$$