Asymptotic normality is about how a sequence of random variables (often estimators) behaves for large sample sizes n. In essence, an estimator $\hat{\theta}_n$ is said to be *asymptotically normal* if, when suitably centered and scaled, it converges in *distribution* to a normal random variable.

Typical Formulation

A common statement of asymptotic normality is:

$$\sqrt{n}\big(\hat{\theta}_n - \theta\big) \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

where

- $\hat{\theta}_n$ is the estimator of some true parameter θ.
- $\xrightarrow{d}$ denotes *convergence in distribution*.
- $\mathcal{N}(0, \sigma^2)$ is a normal distribution with mean 0 and variance σ^2.

The factor $\sqrt{n}$ is typical in many classical asymptotic results such as the Central Limit Theorem (CLT), where variances shrink like $1/n$, but in other problems one might see different scaling factors.

Modes of Convergence

When we say

$$\sqrt{n}\big(\hat{\theta}_n - \theta\big) \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

we specifically mean *convergence in distribution*. This is weaker than convergence in probability or almost sure convergence:

- **Convergence in distribution** ($\xrightarrow{d}$): A sequence of random variables X_n converges in distribution to X if, for every continuity point x of the CDF of X,

  $$\lim_{n \to \infty} P(X_n \le x) \;=\; P(X \le x).$$

  In intuitive terms, their distribution functions get closer and closer.

- **Convergence in probability** ($\xrightarrow{p}$): $X_n \xrightarrow{p} X$ if for every $\varepsilon > 0$,

  $$P\big(|X_n - X| > \varepsilon\big) \;\to\; 0 \quad \text{as } n \to \infty.$$

  Convergence in probability implies that with high probability, X_n is close to X.

- **Almost sure convergence** ($\xrightarrow{\text{a.s.}}$): $X_n \xrightarrow{\text{a.s.}} X$ if

  $$P\big(\lim_{n \to \infty} X_n = X\big) \;=\; 1.$$

  This is the strongest form of convergence: it happens for almost every outcome (except possibly a set of probability zero).

We have the following hierarchy of convergence:

$$\xrightarrow{a.s.} \implies \xrightarrow{p} \implies \xrightarrow{d}$$